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THE FOURTH DIMENSION.*

IS there a fourth dimension? Is there a space near but unknown to us reaching away to boundless distance and in which if we could move we could perform marvellous feats of motion unknown to us in our restricted field of activity? With this question I introduce to you for discussion a problem interesting alike to the accomplished mathematician and layman and which has attracted considerable attention at various times during the past 150 years.

I may bring the matter before you in the following simple manner. If we imagine a point moving away from its position in any direction, it will describe a line, and if the direction of motion be unchanged the line will be straight. Next, if a limited portion of this line be moved perpendicular to itself, it will trace out a surface, and if the direction of motion of the line be invariable, the surface will be plane. Further, if the distance moved be equal to the length of the line, a square will be formed. Then, if the square be moved perpendicular to itself, a distance equal to the length of one side, a cube will be generated. Can we go further with this process of generation of figures by motion of simpler forms? Can we say, *move the cube perpendicular to itself an amount equal to the length of one side and form a new figure?* We know of no such motion, since we have exhausted the possibilities of our space in which only three such motions are possible.

The line generated by the motion of the point is said to have one dimension since it can be measured in one direction. The plane formed after two motions has two dimensions, the cube three. If the fourth motion were possible, the new figure would have four dimensions and the new dimension or direction would be the fourth dimension. The fourth dimension means then a new direction of which we have no knowledge, i.e. a fourth direction or a fourth infinity of positions with its greater freedom.

Strictly speaking, the subject is a branch of mathematics and its serious study requires rather severe mental effort, not-

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*A lecture given before the Mathematics and Physics Club of Queen's University.

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withstanding the fact that Newcomb has called it, rather happily, the "playground of mathematics." It is the fairyland of mathematics, and like other fairylands a land in which we may do all sorts of impossible things.

The serious student of the subject calls it "Hyperspace," but there are comparatively few who are willing to spend the time and effort necessary, so "Hyperspace" never has many in its roll of enthusiastic followers.

Fortunately it is possible to make a rather extended excursion into the land of four dimensions with a very moderate mathematical equipment, as I hope to show, and to find there many things never dreamed of before and also to find *possible* explanations of mysteries that have puzzled for ages. Once the existence of the new direction is granted, very little is impossible.

Without carefully defining what we mean by space, for that might leave us with our ideas more vague than at present, I shall assume that you know what space is better than I can tell you. Probably Lodge's definition, "Room to move about," would satisfy us. We may speak of space of no dimenisons, of one, two, three or of as many as there are independent directions of motion. By independent directions, I mean perpendicular directions, for then, in no way can one be made up of parts that belong to others, as would be the case if they were not perpendicular to each other. For example, a line on the surface of the earth pointing N.E., is pointing E. and N. at the same time. I may explain the dimensions of space as follows: On a straight line, the position of any point is defined by giving its distance from a fixed point of reference; on a plane a point may be located by giving the distances from two intersecting perpendicular lines which we may call axes. Thus two dimensional space requires two distances or magnitudes in order to locate a point. In our space we may assign position to a point by giving distances parallel to three mutually perpendicular lines. We reach this conclusion then, in any space the number of dimensions is the same as the number of magnitudes necessary to locate a point. Again, in space of one direction, there is a single infinity of points spread out along the infinite line. In two dimensions or three dimensions there are a double infinity of points, or a triple infinity of points. Now why do we stop with three dimensional space, for we may easily say, four dimensional space requires four independent magnitudes for specification of the position of a point, or yields a quadruple infinity of points. Or why may we not say, in four dimensional space there are four independent directions?

We may take these two sticks and easily place them perpendicular to each other, then we may take a third stick and place it perpendicular to the other two. These represent the independent directions in two and three dimensional space. Where shall we place the fourth stick for the fourth independent direction? Of course, we can not find such a position, try as we may. Does this mean that there is no such direction? If we believe the results of our experience, assuredly there is no such direction and there is no fourth dimension. Apparently this is the end of the matter. But here is where the interest in the subject begins. The fourth dimensionists, of whom there are a goodly number, and some really earnest writers too, tell us that because we can neither find nor imagine such a direction that its non-existence is not proved. Probably were it not for the fact that a large number of very interesting and some uncanny conclusions follow logically from the assumption of a fourth dimension, we would have heard little of it except in lectures on Hyperspace. But once the existence of the new direction or the fourth perpendicular is admitted, then we are free to go as far as we like. Of course this assumption leads immediately to a geometry of figures in four dimensional space as much more varied in its forms and extensive in its relations than ordinary solid geometry, as solid geometry is more extensive than plane. As in solid geometry all the facts and proofs of plane geometry are valid, so the new geometry of Hyperspace includes all that has gone before in the lower forms.

Since the introduction by Kant of the idea that space might have more dimension than three, which idea was later developed by Gauss and Lobatchewsky, there have been many writers on the subject, some of whom have contributed books and papers of the greatest interest. The great difficulty experienced by everyone is the impossibility of realization of the new direction and the fact that figures can not be visualized, at least by most of us, though we may be able to tell many of their properties. A few people have claimed an ability to think in four dimensions, but in all probability they claim merely a familiarity with figures to the extent that they may make their salient features pass before the mind in rapid succession. After all, that is as much as we may claim for figures in three dimensions.

Now there are two methods by which we may study four dimensional figures and their possible motions. The first method will be plain to anyone with an acquaintance with elementary Analytical Geometry. We may represent the distances of a point from two perpendicular straight lines or axes by the symbols x and y. If the point moves, x and y change, and if the point moves in a definite way, the motion will be represented by a relation between x and y. Thus, if the point describe a circle with radius r about the point of intersection of the axes, the relation between x and y is $x^2+y^2=r^2$, and every other such relation is some form of curve. Two such relations represent two curves in the plane, and, if they are simultaneous, the pair represents the points of intersection of the curves. Thus we may study the relation and properties of lines in the plane by manipulation of equations in x and y. If we turn to three-fold space, we may represent the position of a point by the three distances, x, y and z. If the point moves, x, y, and z change, and if, according to some definite law, there is the corresponding relation amongst x, y, and z. E.g., a point moving on a sphere of radius r and about the origin or intersection of the three axial lines moves subject to the relation $x^2+y^2+z^2=r^2$, and any other relation amongst x, y, and z is represented by a surface in three-fold space. Two such relations when simultaneous, represent the intersections of these surfaces, so three such relations must represent the points of intersection of the three surfaces. So by manipulation of equations in x, y, and z, we may study the properties and relations of figures in three-fold space. In the same way a relation $x^2+y^2+z^2+w^2=r^2$ represents a four-fold figure so situated that every point is at a constant distance from the intersection of the four-directional axes of four-fold space. Two such relations represent a surface, three, a line, and four the points of intersection of the four figures. Then we may study properties and relations of figures in four-fold space by manipulation of equations in x, y, z, and w, and without any reference to the existence or non-existence of a fourth dimension. If it exists, the results given by this four dimensional analysis are true. This aspect of the subject loses no part of its interest because we are unable to say whether or not the possibility of a fourth perpendicular is reasonable. Indeed, we need not concern ourselves with the question of its existence or non-existence.

The second method of study and the one which anyone may follow, is the method of analogy; that is, by comparing and contrasting three dimensional figures with two dimensional, then passing from three dimensional to four. We may gain a deep insight into the geometry of four dimensions by this simple process of analogy, which, while it proves little, still to some extent satisfies the proving sense. We will now carry out this method in one or two simple cases.

Suppose a point is moved a definite distance, say four inches, in a straight line, whose ends may be called terminal points, of which we now have two. As a result we have doubled our number of points and obtained a line. Now let the line move perpendicular to itself, four inches, forming a square. The ends of the lines, or terminal points, have, by their initial and final positions, given us four points, the corners of the square, where the lines meet in pairs. Also, these terminal points have generated two lines, which together with the initial and final position of the moving line, give us four lines or sides. So we now have four lines or sides, four points or corners, and one square. Next let the square move perpendicular to itself four inches, giving us a four-inch cube. The corners of the square or points give us by their initial and final position, eight points or corners, where the lines meet by threes. Again these four corners of the square, by their motion, generate four lines, which with the initial and final positions of the sides of the square, give us twelve lines or edges. Finally, the four lines of the moving square generate four square faces, which, with the initial and final positions of the square, yield us six squares or faces. So we now have one cube, six square faces, twelve lines or edges and eight corners or terminal points. What will we have if we move our cube four inches perpendicular to itself, i.e. in a direction perpendicular at the same time to all of its faces? As before, each corner of the cube by its initial and final position yields sixteen corners or points where by analogy the lines meet by fours. Also, the eight corners by their motion generate eight lines or edges, if we may now call them so. So that we have with the initial and final positions of the edges of the cube, 32 lines. Also these moving lines generate twelve new squares or faces which with the initial and final faces of the cube give twenty-four squares or faces. Then the faces of the moving cube generate 6 cubes, yielding 8 cubes in all. Finally the moving cube generates the tesseract, 4-suareq, hypercube, or cuboid, as it is variously called, the four dimensional counterpart of the cube. The result is—

1 tesseract,
8 bounding cubes,
24 faces,
32 edges,
16 corners.

The following table shows the whole process:

	Points	Lines	Squares	Cubes	Tesseracts
Point	1	0	0	0	0
Line	2	1	0	0	0
Square	4	4	1	0	0
Cube	8	12	6	1	0
Tesseract	16	32	24	8	1

For any number in the table, double the number above and add the number to its left. You will notice what has already been pointed out, that in the plane the lines meet two and two at the corners. In the cube the faces meet in pairs and the lines in threes. In the tesseract, the solids meet in pairs forming planes, the lines by fours at the corner, i.e. at each of the sixteen corners, there are four mutually perpendicular lines. In general we may say that points bound lines; lines, faces; faces, solids; and solids, hypersolids.

We may develop some properties of the hypertetrahedron in much the same way. In a plane we may have three points, each equally distant from each other, and if these are joined we hae the equilateral triangle composed of three points, three lines and one surface. In our 3-fold space we may have four points, each equidistant from all the rest. They lie at the vertices of the regular tetrahedron composed of four points, six lines and four equilateral triangles.

In four dimensional space we may have five points, each equidistant from all the rest, giving us the hypertetrahedron. This is generated from the tetrahedron by drawing lines from the new point to each of the four vertices of the tetrahedron, so we have ten edges in all. Then we may pass a plane through each of the six edges of the tetrahedron and the new vertex giving six new planes or faces, making ten in all. Also from the new vertex we have a tetrahedron resting on each base of the old tetrahedron, so we have five tetrahedrons in all. Collecting the members of this family we have—

1 hypertetrahedron,
5 tetrahedrons,
10 faces
10 lines,
5 points.

We might carry this part of the discussion further, but I will merely add in passing that just as there are five regular polyhedrons in three-fold space, there are six regular hypersolids in four-fold space.

We may note some peculiarities of four-fold space by considering how space of different dimensions can be cut in two parts. Space of one dimension or the straight line can be divided by a point on the line. In order to get from one side to the other, the point must be passed, i.e. the point constitutes a barrier in one dimensional space. In the same way, the straight line constitutes a barrier in two dimensional space, and the plane is a barrier in three-fold space. In each of the foregoing cases, the barrier is a space of one dimension less than the space it divides. Following out the analogy, four-fold space can be separated into two parts only by a complete three-fold space.

Again, in the case of the line divided by the point, the barrier may be passed easily if two dimensional movement is allowed, i.e. we may move out of the line around the point and back again. In the same way a linear barrier in two-fold space is easily passed by recourse to a third direction of motion. Finally, a plane may be passed by recourse to the fourth dimensional motion and a point may be taken around the infinite plane without passing through it.

Suppose we look for a moment at the matter of rotation of a figure in any of these spaces. In one dimensional space no rotation is possible. In a plane, any figure may rotate about any fixed point in the plane, all points of the figure describing concentric circles. In three-fold space, any figure may be rotated about a plane, i.e. an axial plane. To partially elucidate the difficulty we experience here in trying to imagine what is meant by rotation about a plane, we should examine the rotation about an axial line in three-fold space a little more closely. Consider the section of the figure made by a plane perpendicular to the axis. This section rotates about the central point, a point in the axial line. Any other parallel plane section rotates about another point in the axial line. The aggregate of plane parallel section which constitute the entire figure rotate about the aggregate of points, constituting the axial line. So in our four dimensional rotation, each solid section of the four space solid rotates about an axial line and the aggregate of solid sections or the entire figure each rotating about its axial line gives us for the aggregate axis, the axial plane.

I just spoke of the solid section of a four dimensional figure. To make this clearer, let us look at spaces of lower dimensions. The section of a plane figure is made by a line; of a three dimensional figure by a plane, and always gives a plane figure for the section. So by analogy the section of a four-fold figure must be made by space of three dimensions and the result would be a figure of three dimensions. To make this more concrete, the section of a sphere made by a plane is a circle, while the section of a hypersphere made by our space would be a sphere, i.e. if a hypersphere entered our space the section would be a sphere.

So we may speak of the solid sections of other hypersolids. If any such figure could enter our space all we could see would be a three dimensional figure. Take the case of the sphere just cited. Precisely as a sphere entering a plane would be seen in two dimensions as the circle of intersection and would enter suddenly at the tangent point to show an increasing circular size until the meridian section passed, then would shrink up to disappear again, so the hypersphere would appear as a point, the point of contact of the four dimensional figure with three-fold space. Then a small sphere would be seen increasing in size to a maximum, to shrink up and disappear as it came.

We may also study the appearance of the tesseract as it would appear to us. Just as a cube passing through a plane with its face parallel to the plane would show us a square of unvarying size and position, so the tesseract might appear suddenly as a full grown cube, stationary, and of unvarying size until it had passed completely through our space to disappear as suddenly as it came. If it entered in any other direction, the appearance would be different and the figure not stationary. Now all of these representations of hypersolids are sectional and bear the same relation to the figures of which they are sections as plane sections do to solids, i.e. give very incomplete ideas of the figures themselves.

Is there no way in which we may get a perspective view of a hypersolid, just as we may represent a three-fold solid by a drawing on a plane, or by a photograph? Fortunately, there is such. We must expect our representation to be three dimensional and not two and therefore not representable on paper except as a two-fold representation of the three-fold projection. A camera to take photographs of four-fold figures would have to be four-dimensional and the dry plates (?) would be necessarily three dimensional. The representation I will describe I owe to T. P. Hall, who published an article on the subject in "Science" some twenty years ago. It is the only clear description of the method I have seen, but I am confident the method is fairly well known. The figures are copied from the above mentioned paper.

We may approach the problem of representation by projection by treating the simpler case of two-fold projection, and work the analogy out, step by step. The ordinary representation of the cube is shown in Fig. 1.

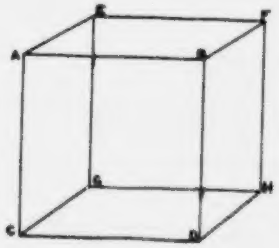


Fig. 1.

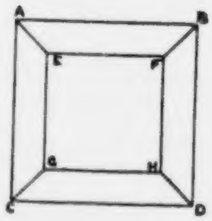


Fig. 3.

If we make such a cube of paper we may unfold it and spread it out on a plane, i.e., develop it, as in Fig. 2.

This figure consists of a central square bounded on each side by another square. To completely close the cube a fourth paper square would be necessary, but the sides of this square

are the lines EF, FG, GH and HE. Thus all parts are accounted for. Now if the eye be held in front of the centre of Fig. 2 and the middle square be pulled forward, with the others falling in behind until they meet, the eye will see the cube as

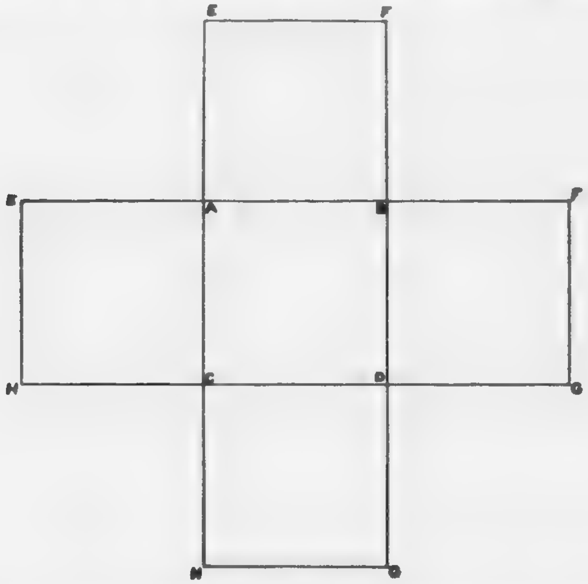
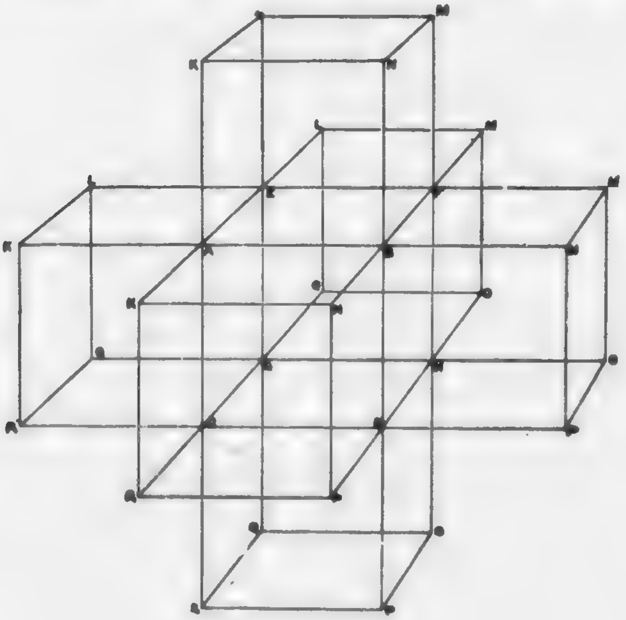


Fig. 2.



in Fig. 3. The various steps may be followed on the diagrams. Thus we have here the central projection of the cube on a plane with the eye opposite the middle point and a short distance in front of the cube.

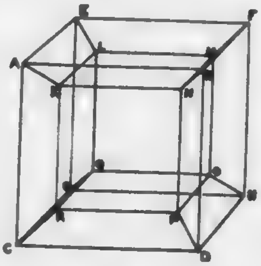


Fig. 5.

Proceeding in exactly the same way, we may study the problem of three-fold representation of a four-fold body, say the tesseract. First, we may spread the tesseract out in three dimensions. By analogy we will do this by taking one of the cubes of the tesseract and building a cube on each face. This corresponds exactly to the development of the cube in two dimensions. The tesseract requires eight cubes according to the table. We have only seven here (Fig.4) but outer faces of the six outlying cubes constitute the faces of the missing cube just as the four lines of the outlying squares in Fig. 2 constituted the sixth square of the cube. Now we must imagine that the eye is a short distance from central cube, but in fourth dimension, just as before we were to imagine the eye in the third. Now take the central cube EFGH and pull it toward the eye and allow the six outer cubes to drop back, swinging about their edges as hinges until they meet, when we will have one cube near the eye, one farther away, and the remaining six joining these, distorted to be sure, but this is a solid perspective (Fig. 5). Step by step we have carried out these two processes exactly alike, and step by step the figures correspond. The outer and inner cubes of Figs. 4 and 5 correspond to the outer and inner squares of Figs. 2 and 4; and the distorted cubes of Fig. 5 to distorted squares of Fig. 3. So we have the appearance in three dimensions of a tesseract viewed from the fourth. The required number of lines, surfaces and points are here in their proper relation and we have a feeling of security over the result. The advantage of this method of projection lies in the fact that all lines, planes and corners can be seen in their relations to each other. What is lacking is the four-dimensional content, just as the volume is lacking in a two-dimensional projection or in a photograph.

One very common method of studying the properties of four-fold figures and one that has been overdone in some respects, is to imagine a race of beings confined to a plane. These would be entirely ignorant of a third dimension and, if we may endow them with powers like our own, we can see how difficult it would be to convince them of the existence of a direction perpendicular to their plane. Some writers seem almost to believe that the existence of the fourth dimension is proved when they have shown the difficulty which a two-flat dweller would experience in attempting to imagine a third dimension. No doubt we may learn much from the study of Flatland and its inhabitants, but we must not take Flatland too seriously.

Let us suppose for a moment that there is a land of plane dwellers and see what we could do to them and how we might mystify them by a few simple three dimensional feats. To one of these beings, two symmetrical triangles could never be made equal however they might be pushed around, but to us who can turn one of them over, nothing is easier. When we turn it over, left becomes right, and right left. So if a four-dimensional motion were possible, symmetrical polyhedrons could be made equal by lifting one into the fourth dimension, turning it over and returning it to its position in three-fold space. Right would have become left, and left, right. A right-hand glove by this rotation would be made into a left-hand glove. A similar change is made by turning the glove wrong side out, but the four-dimensional method would not do so. The effect of turning a person's hand over in four dimensions is interesting to contemplate, and more so, the effect of turning a man around bodily in four dimensions. Everything about him would be inverted; heart, eyes, ears, etc., even to his watch, which would run backwards. There would be no change in the man except in appearance.

Again, consider a ring of some material (?) in two dimensions and a plane dweller placed inside or thrust through an opening in the ring which is subsequently closed. There would be no escape for him unless the wall were pierced. But you or I could reach down and take the languishing prisoner into our three-fold space, carry him across the ring and replace him in the plane, outside his prison. For us, if confined in a room with locked doors and windows, escape would be possible by a four-dimensional motion which would enable us to move away from the room in fourth direction around the walls and back to three-space, but outside the room and with no piercing of walls. To any one watching the performance there would be an instantaneous disappearance and reappearance. The slightest motion into four-fold space means complete disappearance from three-fold space, just as motion from a plane means complete departure from that plane. It has been suggested that collar buttons roll away into the fourth dimension on morning of early classes.

Also, suppose a ring of some flexible material. This could be entered from above, the inside lifted up, pulled over and replaced so that the ring would be inside out. To a plane dweller such a feat is impossible without puncturing the ring. So a hollow rubber ball could be approached from the fourth dimension and the inside wall seized, pulled out, turned over and replaced outside, with the result that the ball has been turned inside out without piercing the wall and with a very little stretching only.

The complete anatomy of a plane dweller would be open to the view of a three dimensional being. So our own interiors would be completely exposed to the view of a four-dimensional being. What a boon to the surgeon! In a very interesting little volume called "Flatland by a Square", by Abbott, a three-dimensional sphere tries to give a poor plane dweller, a square, some ideas of a third dimension. He asks the square to define space. which he does as extension in two directions. These two directions he indicates. The sphere tells of another direction upward. And to see in this direction "upward not northward" the square is told that he would have to have an eye on his side. He says he has one. But the sphere, to make it plainer, says the eye must be on his inside. "An eye in my stomach?" Mr. Square exclaims in horror. But that is it exactly. To see in a fourth dimension you would have to look in a direction which could start from every point in your body, let us say from your stomach, at a place which to a four-dimensional being is a side but to you is an inside. An eye to see in the fourth dimension could not be on the outside of the three-dimensional body.

Now what shall we say as to the existence of the fourth dimension, and what facts if any seem to call for the assumption of its existence The idea that space might prove more than three dimensions seems to have originated with Kant,and was extended by Gauss and Lobatschewsky, but later writers have done much to popularize the subject. The most prolific of these is C. H. Hinton, the author of a number of books and essays. Hinton was a firm believer in the existence of the fourth dimension. He says in one place: "We must be four dimensional, else we could not think about four dimensions," a statement which will find but little acceptance. He says further that, assuming the existence of a fourth dimension, we must have some extensions into this dimension or we must be mere abstractions, just as plane beings must have some existence in a third direction or be abstractions. But until we are sure of the existence of the fourth dimension we need not worry much about our extension or non-extension into it.

A number of people have grasped at the opportunity of the greater freedom afforded by four dimensions and have made this new infinity the abode of departed spirits. The idea is not without interest since the four-dimensional existence would enable those enjoying it to come as near as they liked to mortals in ordinary conditions and remain near without their presence being suspected. Some have gone farther and attached to the phenomena of spiritism a four-dimensional explanation. From this region we are to believe that the crude manifestations of the medium are carried on. Automatic writing, levitation and all the other weird phenomena of the seance might be explained if natural explanations do not suffice. Perhaps these latter should not be neglected before recourse to the fourth dimensional explanation. The experiments of Zöllner with Mrs. Piper, the medium who attracted so much attention a few years ago, have become famous. If two rings be welded together like the lings of a chain, they can not be separated without cutting a link, but if four-dimensional movement were possible, one link could be moved away from the other and returned with the links unsevered but separated. Zöllner gave two such interlinked rings to the medium for separation. It is unnecessary to state that they were not separated. Thus direct experiments have always failed.

Some have attempted to prove the existence of the fourth dimension from the Scriptures, e.g., in Ephesians 3: 18, where St. Paul speaks of the "length, breadth, depth and height," etc., meaning of course four dimensions. Again from Revelations, " I knew a man whether in the body or out of the body I cannot tell," etc., i.e., he passed into four-dimensional space. These people ask "Was the body of Christ after the resurrection four-dimensional, since He appeared to His disciples in such a mysterious manner, and was the Ascension a final disappearance into four-dimensional space"? Were the translation of Enoch and the liberation of Peter from prison phenomena of the same sort?

Certain facts in Chemistry and Physics seem to be partially cleared up by the assumption of the fourth dimension. Certain organic substances seem to require a molecule in which five atoms are arranged in a perfectly symmetrical manner. If they were at the vertices of a hypertetrahedron, the arrangement needed would be available. In Physics, it is well known that certain sugar solutions and specimens of quartz, rotate the plane of polarization to the right and others to the left, otherwise they are the same. The right-handed sugar may be easily changed into the left. If the atom became turned over in a four-dimensional rotation, we might account for this difference. Certain phenomena of electric currents would be clearer if four-dimensional rotation were allowed. Pearson once said that atoms might be *ether squirts,* i.e., places where ether is squirted in from the fourth dimension into our space. Of course the existence of squirts calls for *sinks* where the ether flows out again. The recent developments in radiation phenomena seem to have made this theory untenable. Perhaps Professor Pearsons would call electrons the squirts, but it is hardly likely.

Hinton has made the rather interesting suggestion of the possibility of our own four-fold existence in three-fold space. As we saw earlier, the solid section of a four dimensional figure would be three-dimensional, and we might imagine that our lives are the shadowy three-fold sections of four-fold existence, where form varies from moment to moment as the four-dimensional reality passes through our three-fold space. T. P. Hall made an interesting suggestion in this connection. e says, in substance, "Imagine a dichotomous tree—that is, one that always divides into two branches—passing through a plane; we would have as a plane section, a circle of changing size which would elongate, and divide into two circles, each of which would do the same, etc. This reminds us of the segmentation of cell life observed under the microscope as though this latter was caused by a four-dimensional figure passing through our space."

In closing, may I quote a bit from Hinton's writings on this point? "Were such a thought adopted we should have to imagine some stupendous whole, wherein all that has ever come into being or will come co-exists, which, passing slowly on, leaves in this flickering consciousness of ours, limited to a narrow space and a single moment, a tumultuous record of changes and vicissitudes which are but to us. Change and movement seem as if they were all that existed. But the appearance of them would be due merely to the momentary passing through our consciousness of ever-existing realities."

We should notice that four-fold geometry and four-fold phenomena are different things, and, while we may speak of tesseracts and other hypersolids with safety, it is quite another matter when we talk of natural phenomena in four-dimension, and we must not take some of these matters too seriously. If we admit the existence of a fourth dimension, we should be prepared to go further and allow a fifth, and sixth, and so on without limit. The geometries of these spaces might be studied by the mathematical methods or indeed by the method of analogy, but perhaps four dimensions will give us all the freedom we need.

Some one has said that the fourth dimension is the place of dreams, and I think he said better than he intended, so I am consoled when I think that if this is true, perhaps you have investigated the fourth dimension during this rather long and rambling talk, and that I have not utterly failed as I have tried to tell you of the fourth dimension.

A. L. CLARK.